ON THE SIGNIFICANCE OF SOLVING LINEAR PROGRAMMING PROBLEMS WITH SOME INTEGER VARIABLES

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SUMMARY

Recent proposals by Gomory and others for solving linear programs involving integer-valued variables appear sufficiently promising that it is worthwhile to systematically review and classify problems that can be reduced to this class and thereby solved. Historically, non-linear, non-convex and combinatorial problems are areas where classical mathematics almost always fails. It is therefore significant that the reduction can be made for problems involving multiple dichotomies and k-fold alternatives which include problems with discrete variables, non-linear separable minimizing functions, conditional constraints, global minimum of general concave functions and combinatorial problems such as the fixed charge problem, traveling salesman problem, orthogonal latin square problems, and map coloring problems.

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George B. Dantzig

Recently R. Gomory developed a theory of automatically generating "cutting planes" which permits efficient solution of linear programs in integers in a finite number of steps [1]. This approach has been generalized to a case where some variables are continuous and some are constrained to be integers, by E.M.L. Beale [2], and in a more direct way by Gomory [3], see also [4]. Small scale test problems have been successfully computed. The procedure is so promising that it is relevant to systematically review and classify problems that can be reduced to this class. We shall show that a host of difficult, indeed seemingly impossible problems of a non-linear, non-convex, and combinatorial character are now open for direct attack.

The cutting plane approach was first proposed and its power demonstrated by successfully solving an example of a large scale traveling salesman problem by Fulkerson, Johnson, and the author [5]. Manne and Markowitz explored this technique further in [6] and pointed how it could be applied to solve problems involving non-linear objective forms (separable in the variables but not convex).

In Section I we shall give a general description of the cutting plane approach and then describe the principles for solving several general type problems. In the later sections

these will be applied to several well-known problems. The outline for the paper is as follows:

Section I: General Principles

(a) The Method.

(b) Dichotomies.

(c) k-fold Alternatives.

(d) Selection from many pairs of regions.

(e) Discrete Variable Problems.

(f) Non-Linear Objective Problems.

(g) Conditional Constraints.

(h) Finding a global minimum of a concave function.

Section II: Fixed Charge Problem.

Section III: The Traveling Salesman Problem.

Section IV: The Orthogonal-Latin Square Problem.

Section V: Four-Coloring a Map (if possible).

I: General Principles

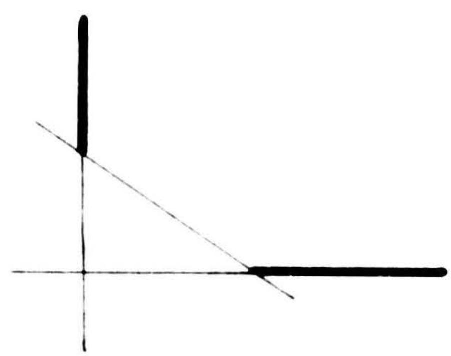
(a) The Method: The cutting plane method consists in first solving the linear programming problem without the integer constraints. If the optimum solution happens to satisfy these conditions all is well. If not then additional linear inequality constraints (called cutting planes) are added to the system in such a way as to remove the non-admissible extreme

point solution and yet retain all admissible solutions (e.g., those having integer values). In principle, that this could be done has been known for sometime. For example, a plane that goes through all neighboring vertices of the non-admissible extreme point can be used as cutting plane in the case where all variables must have values 0 or 1. However such a procedure has been regarded as probably too slow and actual problems until recently were solved using more efficient cutting planes whose validity depended on special arguments. This weakness has been overcome by the recent proposals which generate cutting planes in an efficient manner. It is the author's belief that now it is only a matter of time before a subroutine for integer and partial integer solutions will be part of electronic computer simplex codes.

Let us now turn to the main subject of this paper, types of problems that are reducible to linear programs some or all of whose variables are integer-valued.

Quite often papers will appear in the literature which formulate a problem in L.P. (linear programming) form except for certain side conditions like $x_1 \cdot x_2 = 0$ or the sum of terms of this type such as $x_1 \cdot x_2 + x_3 \cdot x_4 = 0$ which imply for nonnegative variables that at least one variable of each pair must be zero.

Superficially this seems to place the problem in the area of quadratic programming. However the presence of such conditions can entirely change the character of the problem (as we shall see in a moment) and should serve a warning to those who would apply willy-nilly a general non-linear programming method. If we graph the conditions $x_1 \cdot x_2 = 0$, $x_1 \geq 0$, $x_2 \geq 0$, $x_1 + x_2 \geq 1$, the double lines depict the domain of feasible solutions. It will be noted that it has <u>two disconnected parts</u>. If there are



many such <u>dichotomies</u> in a larger problem it can result in a domain of feasible solutions with many disconnected parts or connected non-convex regions. For example, k pairs of variables in which one is zero might lead to 2^k disconnected parts. Usual mathematical approaches can guarantee at best a local optimum solution to such problems, i.e., a solution which is optimum only over some connected convex part.

It has been well known that by special devices that the local optimum solutions could be avoided in many cases by the introduction of integer valued variables but this has only been of passing interest until the recent developments rendered this approach practical. Our purpose here will be to

systematize this knowledge

(b) Dichotomies: Let us begin with the important class of problems that have "either-or" conditions. For such a problem to be difficult computationally there must be many sets of such conditions. Let us focus our attention on one of them, say

(1) EITHER $G(x_1,x_2,\cdots,x_n) \geq 0$

(2) OR $H(x_1,x_2,\cdots,x_n) \geq 0$

must hold for values of $(x_1,x_2,\cdots,x_n)$ chosen over some set S. We do not exclude the case of both holding if possible. For example, a contractor in a bid might stipulate either $x_1 \geq \$10,000$ or $x_1 = 0$. If all bids are nonnegative so that $x_1 \geq 0$, then we can write

EITHER $x_1 - 10,000 \geq 0$

OR $-x_1 \geq 0$.

From other considerations it may be known that no bid can exceed \$1,000,000 so that the set S of interest is $0 \leq x_1 \leq 1,000,000$.

We now assume that lower bounds for the functions G and H are known for all values of $(x_1,x_2,\cdots,x_n)$ in S. If L_G is a lower bound for G and L_H for H then for $\delta = 1$ the condition

(3) $G(x_1,x_2,\cdots,x_n) - \delta L_G \geq 0$

holds for all values of $x_1,x_2,\cdots,x_n$ in S. Similarly for

$\delta = 0$ the condition

(4) $$H(x_1, x_2, \cdots, x_n) - (1 - \delta)L_H \geq 0$$

holds for all values of $(x_1, x_2, \cdots, x_n)$ in S. For our example we would have

$$x_1 - 10,000 - \delta(-10,000) \geq 0$$

$$-x_1 - (1 - \delta)(-1,000,000) \geq 0 \; .$$

The <u>either-or</u> condition (1,2) can now be replaced by

(5) $$G(x_1, x_2, \cdots, x_n) - \delta L_G \geq 0 \qquad (\delta = 0,1)$$

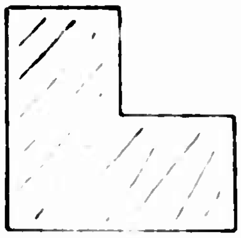
(6) $$H(x_1, x_2, \cdots, x_n) - (1 - \delta)L_H \geq 0$$

(7) $$0 \leq \delta \leq 1$$

where δ is an integer variable. The effect of $\delta = 1$ is to <u>relax</u> the G condition when H holds and of $\delta = 0$ is to <u>relax</u> H when G holds. If G and H are linear functions we have reduced the <u>either-or</u> condition to three simultaneous linear inequalities in which the variable δ must be 0 or 1.

A dichotomy can be used to describe an L-shaped region (non-convex): for example, $x_1 \geq 0$, $x_2 \geq 0$, $x_1 \leq 2$, $x_2 \leq 2$, and <u>either</u> $x_1 \leq 1$ <u>or</u> $x_2 \leq 1$. We replace this by

$$0 \le x_1 \le 1 + \delta$$
$$0 \le x_1 \le 2 - \delta$$
$$0 \le \delta \le 1 \qquad (\delta = 0,1)$$



If now a problem contains not one but several such pairs of dichotomies (1) and (2), each one would be replaced by a simultaneous set (5), (6), (7) in integer variables δ_i.

(c) <u>K-fold Alternatives</u>: More generally suppose we have a set of conditions

(8)
$$\begin{aligned} &G_1(x_1,x_2,\cdots,x_n) \ge 0 \\ &G_2(x_1,x_2,\cdots,x_n) \ge 0 \\ &\vdots \\ &G_p(x_1,x_2,\cdots,x_n) \ge 0 \,. \end{aligned}$$

Suppose a solution is required in which at least k of the conditions must hold simultaneously. We replace this by

(9)
$$\begin{aligned} &G_1(x) - \delta_1 \, L_1 \ge 0 \\ &G_2(x) - \delta_2 \, L_2 \ge 0 \\ &\vdots \\ &G_p(x) - \delta_p \, L_p \ge 0 \end{aligned}$$

where L_1 is the lower bound for $G(x)$ for $x = (x_1,x_2,\cdots,x_n)$

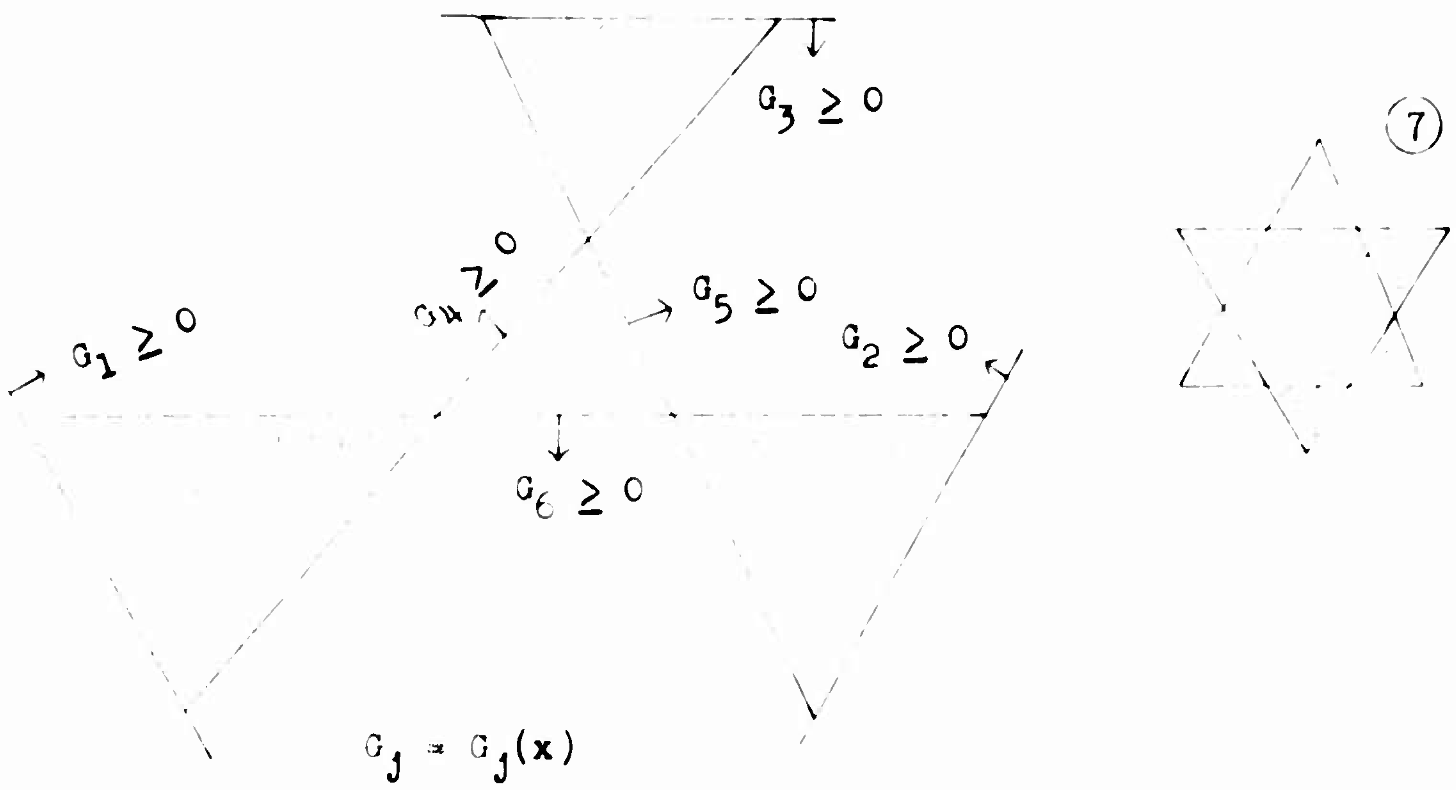
in S and δ_j are integer-valued variables satisfying

(10) $$\delta_1 + \delta_2 + \cdots + \delta_p = p - k$$

(11) $$0 \leq \delta_1 \leq 1.$$

An example of this type of problem might occur if one wishes to find the minimum over the shaded regions described by $G_1 \geq 0$, $G_2 \geq 0$, $G_3 \geq 0$ and at least two of the conditions $G_4 \geq 0$, $G_5 \geq 0$, $G_6 \geq 0$ as in (12).

(12)



(d) Selection from many pairs of regions: The six-pointed "Star of David" region shown on the right in (12) can best be described by a dichotomy in which a point must be taken from one of two triangles. It is only when there are many such pairs to be chosen at the same time that the problem becomes significant. In general we might have several pairs of regions (R_1, R_1'), $(R_2, R_2'), \cdots, (R_n, R_n')$ and the solution point x must lie in either R_i or R_i' for each i. For each pair R and R' we proceed as follows. Let region R be described by a set of inequalities $G_1(x) \geq 0$, $G_2(x) \geq 0, \cdots, G_n(x) \geq 0$ and R' by $H_1(x) \geq 0$, $H_2(x) \geq 0, \cdots, H_n(x) \geq 0$. The condition that the point must be selected from either the first or second region can be written

$$
(13) \qquad \begin{array}{ll}
G_1(x) - \delta L_1 \geq 0 & H_1(x) - (1-\delta)L_1' \geq 0 \\
G_2(x) - \delta L_2 \geq 0 & H_2(x) - (1-\delta)L_2' \geq 0 \\
\vdots & \\
G_m(x) - \delta L_m \geq 0 & H_n(x) - (1-\delta)L_n' \geq 0
\end{array}
$$

$$0 \leq \delta \leq 1, \ (\delta = 0 \text{ or } 1)$$

where L_i, L_i' are lower bounds for G_i and H_i. The more general case of selection from several regions can be done by introducing several δ_i as in (10) and (11).

(e) Discrete Variable Problems: Suppose that a variable is constrained to take one of several values: $x_1 = a_1$ or $x_1 = a_2, \ldots,$ or $x_1 = a_k$ and at the same time several other variables areas are also constrained the same way. It would be a formidable task to test all the combinations. Instead we replace each k-fold dichotomy by

(14) $$x_1 = a_1\delta_1 + a_2\delta_2 + \cdots + a_k\delta_k$$

(15) $$\delta_1 + \delta_2 + \cdots + \delta_k = 1 \qquad \delta_j = 0 \text{ or } 1.$$

Similarly let $x = (x_1, x_2, \cdots, x_n)$ represents a vector which may only take on specified vector values $x = a^1$ <u>or</u> $x = a^2$ <u>or</u> $x = a^3 \cdots$. This may be replaced by

(16) $$x = a^1\delta_1 + a^2\delta_2 + \cdots + a^k\delta_k$$

(17) $$\delta_1 + \delta_2 + \cdots + \delta_k = 1 \qquad \delta_1 = 0 \text{ or } 1.$$

This device permits the replacement of a non-linear function $F_{ij} = F_{ij}(x_j)$ in a system $\Sigma_{j=1}^n F_{ij}(x_j) = 0$ for $(i = 1,2,\cdots,m)$ by a sprinkling of representative values of x_j, say $x_j = x_j^r$ where $r = 1,2,\cdots,k$. In this case the vector is the set of values $(F_{1j}, F_{2j}, \cdots, F_{mj})$ for some value $x_j = x_j^r$.

(f) <u>Non-Linear Objective Problems</u>: Suppose the objective form can be written

(18) $$\sum_{j=1}^{n} \phi_j(x_j) = z(\text{Min})$$

where ϕ_j is non-linear and non-convex. Let each $\phi(x)$ be approximated by a broken line function. These define a set of intervals $i = 1,2,\cdots,k$ of width h_i and slopes s_i for the approximating chords. We now define y_i as the amount of over-lap of the interval from 0 to x with interval i. Then

(19) $$x = y_1 + y_2 + \cdots + y_k$$

and $\phi(x)$ is given approximately by

(20) $$\phi(x) \doteq b_0 + s_1 y_1 + s_2 y_2 + \cdots + s_k y_k$$

where

(21) $$0 \leq y_i \leq h_i \qquad i = 1,2,\cdots,k.$$

In the case of <u>convex</u> ϕ, the procedure is to replace x and $\phi(x)$ by (19) and (20) and conditions (21). Here the slopes are monotonically increasing so that

(22) $$s_1 \leq s_2 \leq \cdots \leq s_k.$$

For a fixed x, $\phi(x)$ would be minimum if y_1 is chosen maximum; then given y_1 maximum, so that y_2 maximum; etc. In other words for the minimizing solution the y_i are the overlap of the i^{th} interval with the interval 0 to x and all is well.

However if $\phi(x)$ is <u>not convex</u> as in (27), then simple replacement of x and $\phi(x)$ would result for fixed x in y_i with smaller slopes being maximized first. In this case the segments that comprise y_i would be disconnected and our approximation for $\phi(x)$ would no longer be valid. In order to avoid this we impose the condition that

$$(23) \qquad \left.\begin{array}{ll} \text{EITHER} & h_i - y_i = 0 \\ \text{OR} & y_{i+1} = 0 \end{array}\right\}$$

which implies that unless y_i is maximum that $y_{i+1} = 0$ and if y_i is maximum then $y_{i+1} > 0$ is possible. We rewrite this condition

$$(24) \qquad \begin{array}{ll} \text{EITHER} & y_i - h_i \geq 0 \\ \text{OR} & -\,y_{i+1} \geq 0 \end{array}$$

and then replace it formally by

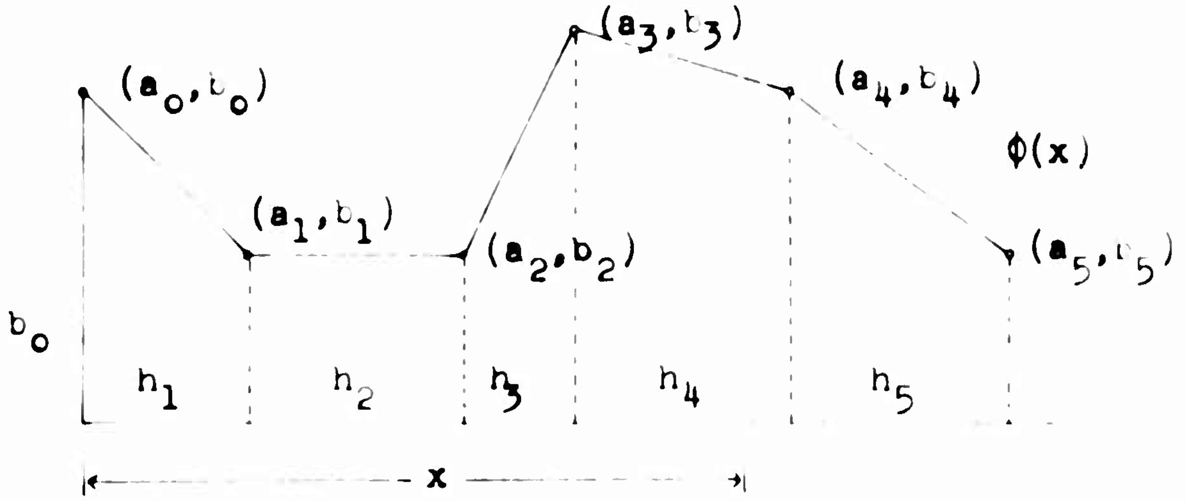
$$(25) \qquad \begin{array}{ll} y_i - h_i - (-h_i)\,\delta_i \geq 0 & i = 1,2,\cdots,k-1. \\ -y_{i+1} - (-h_{i+1})(1-\delta_i) \geq 0 & \\ 0 \leq \delta_i \leq 1 & \delta_i = 0 \text{ or } 1 \end{array}$$

upon substitution of $\delta_i = 1 - \delta_i'$ simplifies to

$$(\ ) \qquad \begin{array}{ll} y_i \geq h_i\,\delta_i' & \\ y_{i+1} \leq h_{i+1}\,\delta_i' & \\ 0 \leq \delta_i' \leq 1 & (\delta_i' = 0,1) \end{array}$$

The above procedure for the non-convex case was discussed in the paper of Manne and Markowitz [6]. The convex case will be found in [7] and [8].

(27)



A second method is worth noting based on (16). Any point on the curve $\phi(x)$ can be represented as a weighted average of two <u>successive</u> breakpoints. Hence we may replace x and $\phi(x)$ by

(28)
$$\begin{aligned} x &= \lambda_0 a_0 + \lambda_1 a_1 + \cdots + \lambda_k a_k \qquad (0 \le \lambda_i \le 1) \\ \phi(x) &= \lambda_0 b_0 + \lambda_1 b_1 + \cdots + \lambda_k b_k \\ 1 &= \lambda_0 + + \lambda_1 + \cdots + \lambda_k \end{aligned}$$

and then impose the conditions that all $\lambda_i = 0$ except for one pair λ_i, λ_{i+1}. For $k = 4$ this may be expressed by

(29)
$$\begin{aligned} \lambda_0 &\le \delta_0 \\ \lambda_1 &\le \delta_0 + \delta_1 \\ \lambda_2 &\le + \delta_1 + \delta_2 \\ \lambda_3 &\le + \delta_2 + \delta_3 \\ \lambda_4 &\le + \delta_3 + \delta_4 \\ \lambda_5 &\le + \delta_4 \end{aligned}$$

where δ_i are integer-valued variables satisfying.

$$(30)\qquad \delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 = 1 \qquad (\delta_i = 0,1).$$

Indeed it will be noted that when $\delta_{i_0} = 1$ for some $i = i_0$ that the inequalities involving λ_{i_0} and λ_{i_0+1} are relaxed but the remainder satisfy $\lambda_i \leq 0$ since their $\delta_i = 0$ by (30).

(_) Conditional Constraints: Suppose x and y are functions of several variables $(x_1, x_2, \ldots, x_n)$ for which upper bounds U_x and lower bounds L_x and L_y are known. We wish to impose conditions such as

$$(31)\qquad x > 0 \Longrightarrow y \geq 0.$$

We can write this as

(32) EITHER $x > 0,\ y \geq 0$
OR $x \leq 0$

which we rewrite as

$$(33)\qquad \begin{aligned} x &\geq \delta L_x \\ y &\geq \delta L_y \\ x &\leq (1-\delta)U_x \qquad (\delta = 0,1) \end{aligned}$$

where the first inequality is written $(\geq)$ instead of $(>)$ because the condition $y \geq 0$ is automatically relaxed for $x = 0$ by selecting $\delta = 1$.

We can now elaborate this to impose conditions such as

(34) $$x > 0 \Longrightarrow u \geq 0$$

$$x < 0 \Longrightarrow w \geq 0$$

which may be written as

(35)
$$\begin{aligned} x &\geq \delta_1 L_x \\ u &\geq \delta_1 L_u \\ x &\leq \delta_2 U_x \\ w &\geq \delta_2 L_w \\ \delta_1 + \delta_2 &= 1 \qquad (\delta_i = 0,1). \end{aligned}$$

For example, suppose in a T-period program we wish to complete a specified work load by the earliest period possible. Let x_t be the cummulative sum of activity levels from the t^{th} period thru the last period T, then we wish to arrange matters so that $x_t = 0$ for the smallest t. In this case we can define for $t = 1, 2, \ldots, T$

(36) $$\delta_t = 0 \Longrightarrow x_t = 0$$

which we may rewrite

(37) $$x_t \leq \delta_t U_t \qquad \delta_t = 0,1$$

where U_t is an upper bound for x_t,

and determine **Min z** where

$$z = \delta_1 + \delta_2 + \cdots + \delta_T \tag{38}$$

<u>Finding a global minimum of a concave function</u>*. Suppose **the concave function** $Z = Z(x_1, x_2, \ldots, x_n)$ **is to be minimized over** a region R. We shall assume R convex for convenience here noting that the devices discussed earlier extend the domain to the wide class expressible by either—or conditions. We suppose R to be given after suitable change in variables in standard linear programming form

$$Rx = e \qquad x \geq 0 \tag{39}$$

where R is a given $m \times n$ matrix and e a given m-component vector.

This is intrinsically a difficult problem because the concave function could have local minima at many, indeed at <u>all</u> the extreme points of R.

The concave function Z may be given **explicitly** or be given implicitly. **For** example, suppose vector y and quantity z for <u>fixed</u> x is given by

$$\begin{aligned} Fy &= f + \bar{F}x \qquad y \geq 0 \\ z &= \alpha x - \operatorname*{Min}_{y \mid x} \beta y \end{aligned} \tag{40}$$

where $\bar{F}$ and F are given matrices and f, α, and β given vectors. This is the situation discussed in the application of these

*This application developed jointly with Philip Wolfe.

methods to Solving Two Move Games with Perfect Information [9]. Here, however, we shall suppose that Z can reasonably be approximated at all points x in R by the minimum Z of a finite set of k tangent hyper-planes.

$$(41) \qquad Z = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - b_i \qquad (i = 1,2,\ldots,k),$$

to the surface $Z = Z(x)$. The problem reduces to choosing Min Z where Z must satisfy at least one of the conditions

$$(42) \qquad \begin{aligned} &Z - [a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - b_1] \geq 0 \\ &Z - [a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - b_2] \geq 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &Z - [a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n - b_k] \geq 0 \end{aligned}$$

which we may rewrite as

$$(43) \qquad \begin{aligned} &Z - [a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n] \geq - M\,\delta_i \qquad (i = 1,2,\ldots,k) \\ &\delta_1 + \delta_2 + \cdots + \delta_k = 1 \qquad (\delta_i = 0 \text{ or } 1) \end{aligned}$$

where $-M$ is some assumed lower bound for the differences; this solution depends on the approximation by k hyper-planes of the function $Z = Z(x)$. The solution given in [9], for the case where Z is given implicitly by (40), requires finding x, y, Min z, and auxiliary variables $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ and $\eta_j \geq 0$ for $j = 1,2,\ldots,n'$ satisfying

(44) $$Ex = e,\ Fy = f + \bar{E}x,\ z = \alpha x - \beta y$$

$$\pi F_j + \eta_j = \beta_j \qquad j = 1,2,\ldots,n'$$

$$\begin{array}{ll} \text{EITHER} & \eta_j \leq 0 \\ \text{OR} & y_j \leq 0 \end{array}$$

where $\pi = (\pi_1, \pi_2, \ldots, \pi_{m'})$ is a row vector, F_j is the j^{th} column of F, β_j the j^{th} component of β.

FIXED CHARGE PROBLEM

Earlier we described a problem where a bidder required that either the order $x = 0$ or $x \geq a$. In this and many other problems there is an underlying notion of a fixed charge that is independent of the size of the order. In this case $x = a$ represents the break-even point to the bidder. In general the cost C is characterized by

(1) $$C = \begin{cases} kx + b & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

where b is the fixed charge. We may write this in the form

(2) $$C = kx + \delta b \qquad (\delta = 0,1)$$

where $x = 0$ if $\delta = 0$ which we impose by

(3) $$x \leq \delta U$$

(4) $$0 \leq \delta \leq 1 \qquad (\delta = 0,1)$$

where U is some upper bound for x. A discussion of the fixed charge problem including this device will be found in the paper by Warren Hirsch and the author [10].

III. THE TRAVELING SALESMAN PROBLEM

We shall give two formulations of this well-known problem. Let $x_{ijt} = 1$ or 0 according to whether the t^{th} directed arc on the route is from node i to node j or not. The conditions

(1) $$\sum_{i,j} x_{ijt} = 1 \qquad t = 1,\ldots,n$$

(2) $$\sum_{j,t} x_{ijt} = 1 \qquad i = 1,\ldots,n$$

(3) $$\sum_{t,i} x_{ijt} = 1 \qquad j = 1,2,\ldots,n$$

(4) $$\sum d_{ij}x_{ij} = z \text{ (Min)}$$

express that (1) there is only one t^{th} directed arc, (2) there is one directed arc leaving node i, (3) there is only one directed arc into node j, (4) the length of the tour is minimum. It is not difficult to see that an integer solution to this system is a tour.

In the paper by Fulkerson, Johnson and the author the case

of a symmetric distance $d_{ij} = d_{ji}$ was formulated with only two indices. Here $x_{ji} = x_{ij} = 1$ or 0 according to whether the route from i to j or from j to i was traversed at some time on a route or not. The conditions

(5) $$\sum_i x_{ij} = 2 \quad , \qquad j = 1,2,\ldots,n$$

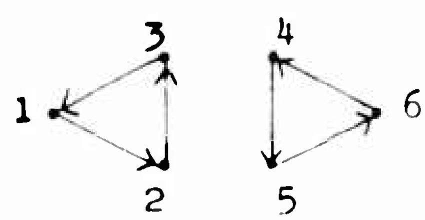
(6) $$\sum d_{ij}x_{ij} = z(\mathrm{Min})$$

express the condition that the sum of the number of entries and departures from each node is two. These conditions are not enough to characterize a tour even though the x_{ij} are restricted to be integers in the interval,

(7) $$0 \leq x_{ij} \leq 1,$$

since sub-tours like

(8)



also satisfy the conditions. However if so-called loop conditions discussed in [5] like

(9) $$x_{12} + x_{23} + x_{31} \leq 2$$

are imposed (in the same manner that cutting planes are introduced as required) these will rule out integer solutions which are not admissible.

IV. THE ORTHOGONAL LATIN SQUARE PROBLEM

A latin square consists of n sets of n objects (1), (2),...,(n) assigned to a n x n square array so that no object is repeated in any row or column. Two latin squares are orthogonal such as

(1)

(1)	(2)	(3)	(2)	(3)	(1)
(2)	(3)	(1)	(1)	(2)	(3)
(3)	(1)	(2)	(3)	(1)	(2)

if the n^2 pairs of corresponding entries are all different. It was conjectured by Euler that there are no orthogonal latin squares for certain n. In spite of a great deal of research by top notch mathematicians the case for n = 10, for example, has never been settled. It has been suggested informally by David Gale that the proposed method be tried in this area.

The formulation is straightforward and well known. Let x_{ijkl} = 0 or 1 according to whether the pair (i,j) is assigned to row k column l or not. The condition that the pair is assigned to only one location is given by

(2) $$\sum_{k,\ell} x_{ijk\ell} = 1 \qquad i,j = 1,2,\ldots,n.$$

The condition that at least one is assigned to each location k, ℓ is:

(3) $$\sum_{i,j} x_{ijk\ell} = 1.$$

The conditions that i, j appear only once in the first and second latin square respectively in column ℓ is given by

(4) $$\sum_{jk} x_{ijk\ell} = 1, \qquad i,\ell = 1,2,\ldots,n.$$

$$\sum_{ik} x_{ijk\ell} = 1, \qquad j,k = 1,2,\ldots,n.$$

Similarly, the conditions that i and j appear only once in the first and second latin square respectively in row k is given by

(5) $$\sum_{j\ell} x_{ijk\ell} = 1, \qquad i,k = 1,2,\ldots,n,$$

$$\sum_{i\ell} x_{ijk\ell} = 1, \qquad j,\ell = 1,2,\ldots,n.$$

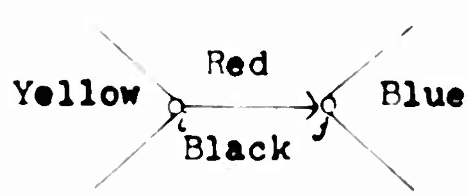
It is interesting to note that every pair of subscripts that are possible out of four are summed to form the six sets of n^2 equations each. For $n = 10$ there are 600 equations, which are too many for a general linear programming code to handle

at the present time. However with some short cuts introduced it might be tractable in the near future.

V. FOUR-COLORING A MAP (if possible)

A famous unsolved problem is to prove or disprove that any map in the plane can be colored using at most four colors where no two regions that have a boundary in common (except a point) have the same color. We shall give two ways to constructively color a particular map if possible. This does not contribute anything to a proof of the truth or falsity of the conjecture except that an efficient way for solving particular problems on an electronic computer may provide a counter example.

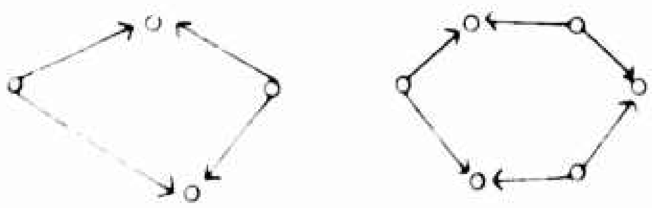
Without difficulty it can be arranged (as below) so



that three regions have at most one point in common which will be called a node. There will be, accordingly, three directed arcs leading from any node i to other nodes j. It is well known that if it possible to four-color a map then (and this will be true conversely) it is possible to treat

the nodes as cities and the arcs as routes between cities and either be able to make a tour of all the cities or to make a group of mutually exclusive sub-tours of the cities in several <u>even</u> (sub-cycle) loops as below.

We may associate with each such even cycle sub-tour, directed arcs that reverse their direction as we pass from node to node



This means the nodes i can be classified into two classes: those which have two arcs pointing away from them and those that have two arcs pointing towards them. Let us set $x_{ij} = 1$ if an arc is part of a sub-tour in the direction of the arrow; otherwise $x_{ij} = 0$. Hence

(1) $$0 \leq x_{ij} \leq 1.$$

It is understood that only arcs (i,j) and variables x_{ij} are considered corresponding to regions that have a boundary in common. All arcs (i,j) that do not correspond to boundaries are omitted in the constraints.

The conditions

(2) $$\sum_j x_{ij} = 2\delta_i \qquad (\delta_i = 0,1)$$

express the fact there must be two arcs on some sub-tour leading away from node i if $\delta_i = 1$, otherwise there are none. The conditions

(3) $$\sum_i x_{ij} = 2 - 2\delta_i$$

state there must be two arcs on some sub-tour leading into node i if $\delta_i = 0$, otherwise none. The three sets of conditions (1), (2), (3) are those of a <u>bounded transportation problem</u> and will be integers (at an extreme point) if δ_i are integers. This would seem to imply that it is only necessary to assume that δ_i are integers and the x_{ij} will come out automatically integral in an extremizing solution without further assumptions. However since the objective form is open to choice by choosing it in a non-degenerate way it is clear that the extreme point solution with integral x_{ij} would be determined by the process.

A second formulation suggested informally by R. Gomory is straightforward. Let the regions be $r = 1,2,\ldots,R$ and let t_r be an integer-valued variable such that

$$0 \leq t_r \leq 3,$$

the four values $t = 0,1,2,3$ corresponding to the four colors. If regions r and s have a boundary in common their colors must be different. Hence for each such pair

(4) $$t_r - t_s \neq 0 \; .$$

This may be written in either-or form:

(5)
$$\text{EITHER} \quad t_r - t_s \geq 1$$
$$\text{OR} \quad t_s - t_r \geq 1$$

which we may rewrite

(6)
$$t_r - t_s \geq 1 - 4\delta_{rs} \qquad \delta_{rs} = 0,1$$
$$t_s - t_r \geq -3 + 4\delta_{rs} \; .$$

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